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Investigating secondary pre-service teachers as teachers and learners of mathematical modeling

Megan H. Wickstrom[†,*] and Elizabeth G. Arnold[‡]
[†]*Department of Mathematical Sciences, Montana State University, Bozeman, MT 59717-2400, USA*
[‡]*Department of Mathematics, Colorado State University, Fort Collins, CO 80524-1874, USA*

[*]*Corresponding author. Email:* megan.wickstrom@montana.edu

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There is a growing attention around the preparation of pre-service teachers (PSTs) as both learners and teachers of mathematical modeling. Current literature focuses on preparing PSTs as learners of modeling, and we extend current work by examining how PSTs integrate their understanding of modeling into the work of teaching. This study follows two groups of PSTs, concurrently enrolled in a modeling course and a secondary methods of teaching course, as they completed a joint unit of instruction in which they designed, refined and enacted a modeling lesson in a secondary classroom. We explore how PSTs build understanding across learning opportunities with respect to four modeling competencies: theoretical, task related, teaching and diagnostic. Our findings indicate that PSTs were able to grow across all four competencies. Theoretical competencies developed from time spent working with modeling tasks as learners and working collaboratively with instructors and peers. Task related and teaching competencies took time to develop as PSTs often wrestled with their conceptions of modeling as learners with existing conceptions of teaching and task design. Diagnostic competencies developed from structured time to reflect on lesson enactment and changes that could be made to best support student learning. Overall, we found that PSTs benefit from engaging in modeling tasks as learners and in enacting components of teaching modeling lessons.

1. Introduction

Over the past decade in the United States, mathematical modeling has gained attention as both a conceptual category and a mathematical practice standard in the Common Core State Standards for K-12 mathematics. Mathematical modeling is the process of understanding authentic, daily experiences through a mathematical lens and is a tool that allows us to make informed decisions through mathematical reasoning. Problems that grow out of authentic experiences rarely come with a prescribed solution strategy, allowing for students to exercise their creativity when working on modeling tasks (Greer *et al.*, 2007; Lehrer & Schauble, 2007). ‘The emphasis on single right answers to questions, being taught a fixed body of specific skills, and reliance on mechanistic algorithms has given students an erroneous view of mathematics' (*GAIMME*, 2019, p. 61), and engaging students in mathematical modeling expands their views of mathematics, empowers them as knowers and doers of mathematics, and teaches them empathetic critical thinking skills. Many 'real-world' problems in textbooks focus on specific mathematical skills where the context is irrelevant, rather than focusing on fully understanding the situation from a decision-making perspective, exploring the benefits and limitations of different modelers' approaches (Dowling, 1999). Modeling embraces the multifaceted nature of mathematics and allows for connections across disciplines, the world and important student issues, which provides opportunities to promote productive mathematical dispositions and foster agency and ownership. Modeling tasks can be leveraged so students explore topics that are meaningful to them, such as social justice issues in their communities, and consider multiple perspectives and ways to take action (Anhalt *et al.*, 2017; Cirillo *et al.*, 2016; Turner *et al.*, 2018).

Although modeling has become more central to the teaching and learning of school mathematics, a dearth of modeling lessons is being implemented in classrooms (Blum & Borromeo Ferri, 2009). Supporting students in the modeling process is challenging for teachers (Blum & Borromeo Ferri, 2009; English, 2009); it involves knowledge of the modeling process, related pedagogical content knowledge (Blum, 2011, 2015; Doerr & Lesh, 2011), students' school-related and lived experiences and facilitating classroom learning and discussion (Zawojewski *et al.*, 2003). To foster mathematical modeling in classrooms, pre-service teachers (PSTs), in particular, need opportunities for learning and teaching modeling.

Researchers have focused on the importance of engaging PSTs in the modeling process as modelers themselves (Anhalt & Cortez, 2016; Cai *et al.*, 2014; Niss *et al.*, 2007), but few have investigated how to support PSTs in teaching modeling (Zbiek, 2016). To our knowledge, no studies have examined PSTs as *both* learners and teachers of mathematical modeling. For this study, we developed and implemented a joint unit of instruction, across a modeling course and a secondary methods of teaching course, to simultaneously engage secondary PSTs as both learners and teachers of modeling. The PSTs in our study were immersed in the modeling process as learners and then provided the opportunity to practice the teaching of modeling as they designed and implemented a modeling lesson in a local secondary classroom. Our study pushes the field forward by examining how PSTs transition their understanding of modeling into teaching, specifically how they reconcile their understanding of modeling as a learner with the realities of teaching modeling.

2. Literature review

2.1. *Defining mathematical modeling and the modeling process*

The phrase, mathematical modeling, has taken on several meanings from using physical models to illustrate mathematics to teaching practices (e.g., NGA, 2010; Cirillo *et al.*, 2016). When we say mathematical modeling, we mean students using their mathematical skills to understand an authentic situation that is based in their daily experiences. Modeling is a process that begins and ends with an authentic situation where students use mathematical tools to represent, understand and solve real problems (Lesh & Doerr, 2003). Borromeo Ferri (2006) describes six components of a modeling process, and we interpret 'modeling' as engagement in this entire process rather than just the creation of the mathematical model itself.

- **Examining** the situation and setting up a problem to be solved.
- **Identifying** variables in the situation and selecting those that are essential.

- **Creating** a model that best describes the relationships among the variables using geometric, graphical, tabular, algebraic or statistical representations.
- **Formulating** conclusions.
- **Interpreting** the results for accuracy and relevance.
- **Refining** the model through validating its potential to account for all relevant variables.

Mathematical modeling has also taken on different perspectives that are characterized by how one views the modeling process, the modeling task and the purpose of model (Kaiser & Sriraman, 2006; Abassian *et al.*, 2019). We view modeling from the 'realistic' perspective, which suggests that modelers work through 'realistic, authentic, and messy tasks' (Abassian *et al.*, 2019, p. 3). The primary goal of the realistic modeling perspective is to support modelers' ability to learn and apply the modeling process to understand and solve authentic problems. There is an emphasis on promoting modeling competencies that are the 'abilities needed to carry out a modeling process' (Kaiser, 2017) and includes both pedagogical abilities and heuristic strategies. We chose this perspective because it mirrors the ways that students will use mathematics in their daily lives to understand an authentic situation. From our perspective, authenticity means that there is no one 'prescribed' solution to the task. Rather, different valid and equally good solutions can be reached based on the assumptions the modeler makes and the mathematics she chooses to employ.

2.2. *Pre-service teachers as learners of modeling*

Researchers agree that PSTs must engage in the modeling process as learners before they can begin to understand teaching mathematical modeling (Niss et al., 2007; Cai et al., 2014; Anhalt & Cortez, 2016). By engaging in modeling tasks as learners, PSTs begin to understand the nuances in the modeling process that set it apart from other mathematical activities. As learners of modeling, PSTs develop an understanding of the components of the process (Doerr, 2007; Zbiek, 2016), build mathematical skills (Doerr, 2007; Anhalt & Cortez, 2016; Zbiek, 2016) and understand related modeling competencies (Gould, 2013; Anhalt & Cortez, 2016; Zbiek, 2016). Doerr (2007) found that PSTs initially viewed the modeling process as linear but, with experience, this perception shifted to understanding the cyclic nature of modeling. Engaging in modeling tasks also helps PSTs understand components of the modeling process. For example, making and recognizing assumptions can be challenging for PSTs; they do not always recognize that every modeling problem involves assumptions (Widjaja, 2013; Zbiek, 2016). Mathematics is often framed as objective, and by examining different assumptions, PSTs may realize that the perspective of the person answering the question matters in developing a model and a solution. Lastly, PSTs often find validating the model and examining its limitations to be challenging because the mathematical problems they have encountered so far are typically viewed from the perspective of correct or incorrect (Bokova-Güzel, 2011). Modeling tasks challenge PSTs' perceptions of what it means to do mathematics and helps build an understanding of both the modeling process and learning opportunities afforded by mathematical modeling.

2.3. *Pre-service teachers as teachers of modeling*

After PSTs have experienced modeling as learners, they must also begin to merge their knowledge of modeling into their existing understanding of teaching. To effectively implement modeling tasks, Stohlmann *et al.* (2015) state that teachers need to have 'robust knowledge of the content in those tasks, possible models that students may develop, the difficulties that students may encounter during the process, and knowledge of pedagogical practices that focus on student-centered discourse' (p. 22). Specifically, PSTs need experiences in (1) choosing appropriate modeling tasks for students; (2) anticipating how students' models might develop over the course of several lessons; (3) understanding multiple approaches to a task; (4) selecting appropriate tools and materials to develop the model; and (5) devising strategies for students to evaluate their models (Doerr, 2007, p. 70).

Further, mathematical modeling is inherently collaborative; modeling allows for multiple approaches and mathematical perspectives, and teachers and their students must work together when developing and revising their mathematical models. When teaching modeling, the teacher must feel comfortable acting as a facilitator, guiding their students as they engage in the modeling process. Because of this, it is important for PSTs to learn pedagogical strategies that support inquiry-based teaching and provide students encouragement and guidance during the modeling process (*GAIMME*, 2019).

2.4. Mathematical modeling and teaching competencies

Just as competencies exist for the learning of mathematical modeling, we can also view the teaching of mathematical modeling through different interconnected competencies. In considering the teaching of mathematical modeling, Borromeo Ferri and Blum (2010, p. 2047) outlined four modeling competencies for PSTs:

1. Theoretical competency (knowledge about modelling cycles, about goals/perspectives for modelling and about types of modelling tasks).
2. Task-related competency (ability to solve, analyze, and create modelling tasks).
3. Teaching competency (ability to plan and perform modelling lessons and knowledge of appropriate interventions during pupils' modelling processes).
4. Diagnostic competency (ability to identify phases in pupils' modelling processes and to diagnose pupils' difficulties during such processes).

We view these four competencies as interconnected and in support of one another. Theoretical and task-related competencies are deeply connected to mathematical content knowledge and knowledge of mathematical modeling as a discipline. PSTs must engage in modeling tasks as a learner to understand the modeling cycle and the ways to approach a modeling task. Teaching and diagnostic competencies are related to knowledge of content and students and knowledge of content and teaching. When enacting a modeling task, the teacher must be aware of several pathways through the task and how to support and connect students' strategies in relation to the content at hand.

3. Theoretical underpinnings and research questions

Much of the work of teaching mathematical modeling occurs when teachers are developing modeling tasks, anticipating students' strategies, enacting the task in a lesson and reflecting on their experiences (Carlson *et al.*, 2016). As mathematics teacher educators, we believe that PSTs develop knowledge needed to teach mathematical modeling through practice. This aligns with the theoretical construct of Pragmatism (Hammond, 2013), the idea that knowledge bases are fluid and shaped by interactions with others and the environment. Hammond (2013, p. 613) states, 'pragmatism tells us that what we know is provisional and arrived at through a transaction between agent and environment.' Teaching is a generative process in that teachers learn through interactions in the classroom along with purposeful reflection. Each day, as teachers create new knowledge, their reality and understanding of their classroom changes and subsequently creates new questions to investigate.

The process of mathematical modeling, both from the perspective of the teacher and of the learner, is fluid. The teacher reconciles her understanding of modeling as a learner with task development and instructional planning. She also balances openness in determining how to provide guidance to students during the task. She makes choices on how to respond and share students' strategies. No matter how much she plans for the task, she gains invaluable information on implementation through teaching and interactions with students. PSTs need time to practice and develop skills for teaching modeling; this includes engaging with modeling competencies independently and collectively, collaborating with instructors and peers and implementing and reflecting on modeling tasks developed for the classroom.

The theoretical lens of Pragmatism and the four modeling competencies outlined by Borromeo Ferri and Blum (2010) guided us in the design of the study, the experiences we provided PSTs in our unit of instruction and in data analysis. Within and across our data we investigated the following questions:

1. How does PSTs' knowledge of teaching modeling develop as they design and implement a modeling task?
   - What modeling competencies are PSTs developing and what experiences aid in this development?
   - What modeling competencies need greater attention and what changes could be made to the instructional unit to address this need?
2. How do interactions with instructors, peers and students shape PSTs' knowledge of teaching modeling?
3. Overall, what experiences can be added to existing coursework to prepare PSTs as teachers of modeling?

4. Methods

4.1. Context of the study and study participants

Secondary PSTs in a Rocky Mountain university were concurrently enrolled in two courses: (1) Mathematical Modeling for Teachers (MMT) and (2) Methods of Teaching Secondary School Mathematics (MTSSM). PSTs had several semesters of practicum experience prior to these courses and were preparing to take over as student teachers in local classrooms the following semester. The first author was the instructor of MTSSM and the second author taught MMT.

MMT is a course focused on the modeling process, and the instructor engaged PSTs in mathematical models that are accessible to secondary school students. PSTs engaged in problem solving, communicating convincing mathematical arguments to support models, connecting mathematics to numerous real-world domains, representing mathematical and physical phenomena and applying mathematical reasoning (algebraic, probabilistic, geometric, analytic and statistical). Emphasis on the use of technology and simulation was present throughout the semester.

MTSSM is a methods course where PSTs engaged in research-based instructional strategies, methods for teaching and assessment and curriculum design and planning; this included discussing how mathematical modeling fits in the K-12 mathematics curriculum and discussion of modeling as a mathematical practice. The course was designed to enable PSTs to become effective professional decision makers in the context of standards-based teaching. PSTs engaged in hands-on activities including designing tasks, lessons and assessments.

In this study, we focus on two groups of three PSTs. There were three groups total but we chose to focus on these two groups because the modeling tasks they developed allowed for variation and choice whereas the third group developed a modeling task more aligned with a word/application problem. The first group developed a task relating a femur bone to height and the other group developed a task investigating the best ski pass to purchase for the season. These modeling tasks are described later in the paper, but we refer to these groups as Femur Bone and Ski Pass. When we utilize the word ‘modeling task’, we are referring to the modeling question the group posed, and when we utilize the world ‘modeling lesson’, we are referring to the task in conjunction with pedagogical practices.

4.2. *Unit of instruction*

To combine the work of learning modeling with the work of learning to teach modeling, we co-developed and implemented a unit of instruction midway through the semester in our respective courses. This unit provided PSTs an opportunity to apply what they learned about modeling in local secondary classrooms. We were well-situated to collaborate on this project in that both of us are modeling researchers and have worked with in-service teachers on modeling tasks. Our situation is unique in that most universities do not have a dedicated modeling course, but the ideas we share could be integrated into either existing practicum coursework or mathematics content coursework focused on modeling.

The unit included multiple activities, spanning four weeks that supported PSTs’ growth as both teachers and learners of modeling (Fig. 1). PSTs worked in teams of three to develop and implement a modeling lesson in a local secondary mathematics classroom, across two 50-minute class periods. We pre-determined the teams and wanted to make sure PSTs had a strong support network to design and enact their lessons and their audience of secondary school students would be receptive. Thus, we contacted local cooperating teachers (CTs) who were familiar with modeling, regularly incorporated inquiry-based instructional practices in their classrooms and were willing to give PSTs feedback on their modeling task.

In week one, PSTs designed a modeling task and built solutions for it as part of their coursework in the MMT course. PSTs could draw from pre-existing tasks, but we encouraged them to consider authentic situations that were relevant to the local students. Following task design, their peers worked through the task, built solutions and offered feedback, which PSTs then used to revise their task.

Next, during a week in MTSSM, PSTs embedded the modeling task into a lesson, noting how school students might respond and including relevant teaching moves (i.e., questions, prompts to elicit students’ thinking, etc.) to engage students in the modeling process. PSTs implemented a challenging portion of their task with peers, received feedback and, again, revised their lesson accordingly. They also sent their lesson plan to their CT for feedback.

The third week of the assignment focused on enacting the lesson in a secondary school classroom with approximately 25 students. The CT was present in the room but PSTs, as a group, took on full teaching responsibilities. After the lesson, the CT provided verbal feedback to PSTs and us about the PSTs’ enactment of the lesson. In the fourth week, PSTs reflected on their enactment, describing expected and unexpected events, the extent to which school students were engaged and changes to improve the task and lesson. Lastly, we asked PSTs to create a 20-minute presentation, shared publicly with members and students of the department at the university, summarizing their experiences.

4.3. *Data collection*

We collected multiple forms of data, including PSTs’ modeling lesson plans, their reflections and their responses to a final exam question describing the process of mathematical modeling.

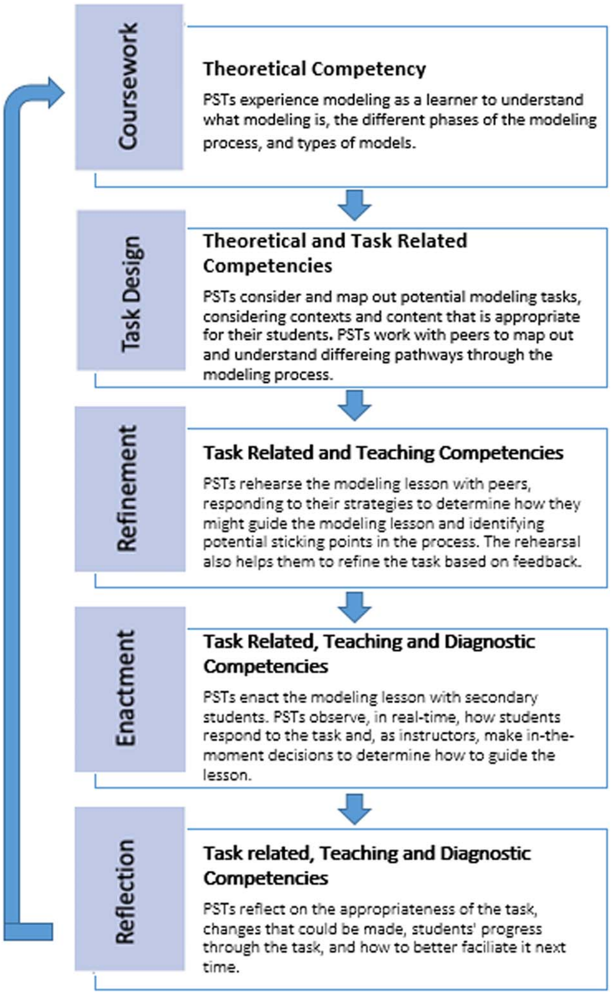

FIG. 1. Phases of design, enactment and reflections in the process of developing and implementing a modeling task.

4.4. *Modeling lesson plan*

PSTs created a lesson plan consisting of several required components. PSTs first stated the learning goals, objectives, content standards, mathematical practices and required materials for the lesson. Next, they outlined the teacher's actions and the anticipated students' actions and separated these actions into a launch, exploration and closure. Lastly, they included opportunities to differentiate the lesson and formatively assess students. These lesson plans were a major source of data as they illustrated PSTs' anticipated instructional choices and components of the modeling process.

4.4.1. Preliminary reflection on the modeling task. After PSTs received feedback from their peers and CTs about their modeling task, each PST responded to the following prompts in a two-page written reflection:

- How did the group build a solution to the model? Was it in a way that you expected and, if not, what could you change to get at the mathematics you wanted?
- How, if at all, will you respond to and incorporate the groups' feedback?
- What other mathematical approaches might students use to solve the task? How will you connect and build off different mathematical approaches?

These preliminary reflections illustrated how PSTs developed their modeling task and the decisions they incorporated in their revision.

4.4.2. Final reflection on the modeling lesson. After PSTs implemented their lesson, they reflected on the lesson 'as is' and 'as it could be'. In reflecting on 'as is', PSTs discussed what went well, what did not work, what the students did that was unexpected, any alterations they made to the lesson, to what extent students were engaged in the process and how they did (or did not) support the students' opportunities to engage in the modeling process. In reflecting on 'as it could be', PSTs discussed changes and how they will apply what they learned to the development and enactment of future modeling lessons. These final reflections also highlighted the components of the modeling process that stood out to them.

4.4.3. Final exam modeling question. During the final exam in MTSSM, PSTs responded to the following prompts: (1) How would you describe modeling? (2) What were the important processes? and (3) What were the challenging processes? These data illuminated the components of the modeling process PSTs thought were necessary, providing a justification to why they made some of the decisions they did when developing and enacting their modeling lesson.

4.5. Data analysis

Data analysis occurred in several phases. Across each phase, we sought to understand how PSTs' knowledge about mathematical modeling as both a learner and as a teacher was developed, co-constructed, refined and integrated into practice as they engaged in the unit of instruction.

First, we created a descriptive timeline composed of three phases to group our data: Development, Enactment and Reflection (Carlson *et al.*, 2016). In each phase we considered the following artifacts and guiding questions as shown in Fig. 2.

Next, we considered modeling attributes emphasized in the MMT and MTSSM courses. We each independently constructed a list of attributes within each of the four modeling competencies and then met to discuss the attributes that would become codes (Fig. 3). Many of the codes, especially in the Enactment phase, describe all four competencies simultaneously and illustrate that modeling knowledge for teaching is interconnected.

For the Development and Enactment phases, we read the lesson plans and reflections independently and used our attributes as a coding scheme. Using a descriptive coding process (Miles *et al.*, 2014), we identified where codes appeared in the text and wrote a brief summary of the context. We discussed our initial coding to create shared consensus and discussed ways in which PSTs adhered to modeling attributes and also ways in which they faced difficulties. We utilized the Reflection phase as a form of triangulation between what we observed as researchers in the Development and Enactment phases and what PSTs self-reported in the Reflection phase to identify what PSTs were aware of during implementation. After data analysis, we revisited the codes in relation to each of the modeling competencies to determine how each code could inform our overall understanding of how PSTs develop as both learners and teachers of modeling and what experiences are fundamental to this development.

	Development	Enactment	Reflection
Data Sources	• PSTs' preliminary reflections on task development	• PSTs' lesson plans • PSTs' final reflections on lesson enactment	• PSTs' final reflections on lesson enactment • PSTs' modeling descriptions from final exam
Guiding Questions	• How and why did PSTs modify their initial task across the design phase? • What attributes did PSTs attend to (or not) during the design phase?	• How did the PSTs describe the enactment of the modeling lesson? • How did PSTs incorporate (or not incorporate) modeling attributes into enactment?	• What did PSTs identify as important attributes of modeling? • What did PSTs identify as points of difficulty in teaching the modeling process?

FIG. 2. Artifacts and guiding questions for data analysis.

CODES	COMPETENCIES: *Theoretical*	*Task Related*	*Teaching*	*Diagnostic*
Development				
Real World Connection	✓	✓		
Connection to Appropriate Content Standards		✓	✓	
Incorporation of Modeling Process	✓	✓		
Rich Mathematics and Opportunities for Creativity	✓	✓		
Enactment				
Relevance to Students		✓	✓	✓
Opportunities for Student Engagement			✓	✓
Openness and Opportunities for Defining Mathematics	✓	✓	✓	✓
Consideration of Constraints and Variables	✓	✓	✓	✓
Multiple Solutions to the Task	✓	✓	✓	✓
Opportunities for Model Revision	✓	✓	✓	✓
Reflection				
Attention to Student Strategies			✓	✓
Awareness of Potential Revisions to Lesson	✓	✓	✓	✓

FIG. 3. Codes with respect to the three phases and four modeling competencies.

5. Development, Enactment and Reflections

In this section, we describe each groups' Development, Enactment and Reflection phases of their modeling tasks.

Original Modeling Question: You are an archeologist on a dig in (insert some interesting location here)! You recently found a femur bone (or skull) and you are on the mission to figure out how tall the person was.

FIG. 4. Femur Bone group's original modeling question.

5.1. *Femur bone investigation*

5.1.1. Task development. The Femur Bone task draws on linear relationships and lines of best fit. PSTs wanted students to be able to collect and interpret data in the classroom, so they looked for naturally occurring linear relationships. They searched online and found that the human femur bone is about a quarter of a person's height and the circumference of the human skull is about a third of a person's height. PSTs tested this assumption by measuring both their femur bone and skull and found that the femur bone relationship seemed more reliable. They brainstormed when and where people might care about bones and started with the following modeling question, as shown in Fig. 4.

Following construction of the initial task, PSTs gave it to their peers and encountered several realizations. They discovered the task could benefit from additional background knowledge. Many of their peers were not sure where the femur bone begins or ends and needed clarification when measuring, so PSTs instructed their peers on how to measure. Their peers also suggested that providing a meaningful context (such as a location and the length of the femur bone that was found) would be helpful in providing purpose. To address these concerns, the instructing PSTs found that a room had recently been discovered in the Pyramid of Giza and connected this world news to their task.

Once they started on the mathematics of the task, PSTs noticed their peers measuring femurs to determine a ratio, which was not the mathematical direction they hoped to go. PSTs wanted students to graph multiple points and determine a line of best fit so that they could have conversations about linear relationships, variability and assumptions made about the relationship between height (i.e., dependent variable) and femur length (i.e., independent variable). To reorient toward graphing, the instructing PSTs pivoted and asked the entire class to measure and report what they found in a table. The instructing PSTs noted that in the secondary school classroom they should emphasize collecting a large dataset so that students do not rely on three or four data points to calculate a line of best fit and to predict the height of the person based on the length of their femur bone. In particular, PSTs wrestled with the idea of giving students an actual measurement for the femur bone because they were afraid that students would find a classmate with a similar size femur and use their height rather than construct a model to make a prediction. One PST stated,

> As the groups completed our modeling task, I realized that our group needs to consider when to give the students the actual length of the femur bone they found. If we give them the lengths at the beginning of the task, our fear is that students will try and find a correlation using just this one length, and once they find someone who has a similar length femur bone, they will be 'done'.

After they oriented their peers toward graphing, the instructing PSTs noted that different groups generated different lines of best fit. Some groups noted y as the length of the femur and x as the height while others noted the opposite causing the 1 to 4 relationship to appear differently. One of the instructing PSTs, Anne, reflected on this in the following way:

> One group made the equation $\hat{y} = 0.277x$, where x is the [person's] height and y is the [length of the] femur while the other group did $\hat{y} = 2.27x + 34$ where x is the length of the femur and y is

Final Modeling Question:

You are an archeologist on a dig into the new room in the Great Pyramid. You recently found a femur bone, can you predict the height of the person it belonged to?

FIG. 5. Femur Bone group's final modeling question.

> the [person's] height. Both could be valid models but look very different. The first model you could see the $\frac{1}{4}$ relationship emerging where the second model did not make it to that point (if it was a $\frac{1}{4}$ relationship, then the slope should be close to 4). I didn't expect both groups to come up with very different equations.

The instructing PSTs noted that to focus students in on the 1 to 4 relationship they would restrict to one or two equations together as a class. Heading into instruction, PSTs found comments from their peers both helpful and conflicting, primarily with regard to the openness of the task and the focus on the mathematical content. One PST, Margot, discussed that it was challenging finding a balance between navigating content and keeping the task open for students to explore. Her perceptions of teaching mathematics involved addressing particular content standards (i.e., line of best fit and variability) and it was challenging to merge with the openness involved in modeling.

PSTs decided to narrow the task but add brainstorming questions at the beginning of their lesson to generate student interest. In particular, the PSTs changed the question to predicting a height rather than determining a set height. After receiving feedback from their peers, the instructing PSTs refined their modeling question (Fig. 5).

5.1.2. Lesson enactment. PSTs motivated the lesson (see Appendix for an excerpt of the lesson) with a news clip of the discovery of a new room in Giza. They asked students to discuss what tools and data would be helpful in finding a solution. After brainstorming, students shared out ideas like measuring their own femurs and comparing the femur to other full skeletons on the dig site to determine if a relationship exists. PSTs asked students to measure their own heights and femurs and recorded the data on the board so students could look for patterns. After the data was displayed on a graph, the class discussed if a relationship existed and what it might be.

In reflecting on the lesson, the instructing PSTs stated the launch of the activity went well: 'Students came up with a variety of great questions, good ideas for tools, and helpful data that they wanted. They came up with many contextual questions and assumptions (that we hadn't even thought of).' Similar to the work with their peers, the instructing PSTs noted that several groups went toward finding a simple ratio and were done rather quickly. In addition, the instructing PSTs felt that the students did not have the pre-requisite knowledge they had expected for this lesson, and it was challenging for students to move from ratios to interpreting data in context and finding a line of best fit. They stated, 'Unfortunately, students didn't find slope to create a linear relationship between the femur lengths and heights of their classmates. Students knew less than we anticipated about the relationship.' In the future, to help students make the connection between ratio and slope in the context of the graph, the instructing PSTs described how they would allow for more time for students to understand and contextualize the task, introduce technology, and build connections between ratio and slope. They stated:

> The first day would be used to introduce modeling tasks to students, familiarize them with the task and contextual factors, and have a mathematical discussion about the ratio they find. The second day

of this lesson would be spent exploring the relationship between the ratio, rate of change, and linear relationships.

The instructing PSTs also acknowledged that they were somewhat underprepared when they enacted the lesson in that they did not anticipate all the possible directions students might go, even though they encountered multiple strategies during the Development phase. There was a significant jump between understanding ratio and slope in the context of linear functions and applying this knowledge to interpret a dataset. They discussed that in the future they want to be prepared to support students in more than just a single strategy and allow students the freedom to explore a strategy that they did not prepare for.

5.1.3. Reflection. In their reflections, all PSTs in this group described modeling as a process to solve a real-world problem using mathematics and that there should be back and forth between the mathematical problem and the real-world. All three also mentioned the 'ideal' modeling experience has no limits or bounds with no correct answer and no two models should ever be the same because modelers bring their personal knowledge to the situation. Only one of the PSTs discussed the importance of testing the model and revising it.

When considering modeling from a teaching perspective, PSTs primarily discussed grappling with openness, stemming from students' multiple perspectives. All three discussed that limits must be set to achieve particular content goals in the classroom, and this can be particularly challenging for students. One PST, Margot, stated,

> When implementing a mathematical modeling task in a [secondary] school classroom it is important to consider students' prior experience. Tasks can be overwhelming and stressful to students who are used to 'one right answer'. Start by giving tasks with more structure, and then slowly start to take away the guidance and structure so that students become used to the openness.

All three PSTs discussed how teachers need to be prepared to navigate openness and anticipate how students are going to work through the modeling process. This allows for the opportunity to support students and find related materials they may ask for. Margot described this idea in the following way:

> It is also important to consider (beforehand) the various strategies and directions that students will use so that the teacher can be prepared to support students. If a teacher does not consider the ways students might approach the task, then they might not have the necessary information.

PSTs also discussed that the authenticity of the modeling task helps engage students in the challenging aspects of modeling. Lastly, Anne noted that part of the work of teaching modeling is dealing with uncertainty. She stated, 'keep in mind, there is no way to prepare for all the answers or questions the students come up with.'

5.2. Ski Pass investigation

5.2.1. Task development. The Ski Pass task draws on content related to systems of equations. Textbook problems related to systems of equations are often boiled down to choosing the best cell phone plan or gym membership, but the instructing PSTs wanted students to investigate a situation with greater relevance. With the proximity of many local ski hills, they thought skiing might be of interest to students. PSTs started with the following modeling question (Fig. 6).

The instructing PSTs purposely designed this question to be open-ended with several assumptions the modelers could explore such as how many times the individual decides to ski for the season, the length

Original Modeling Question:
If you're going to go skiing this weekend, what is the best ticket option and why?

FIG. 6. Ski Pass group's original modeling question.

of the season, the age of the skier, whether the individual owns ski equipment, whether the person is on vacation or a local, and if the skier plans to eat at the resort. The group predicted that students would primarily explore and make assumptions about the number of days they wanted to ski during the season, but they thought students could also add an additional variable in the model, such as food cost or rental fees, and examine the problem from different perspectives. Their primary goal was for students to use linear equations to represent the different types of ski pass tickets and then compare which type of ski pass is most cost efficient.

Upon piloting this task, PSTs were surprised by the lack of progress made by their peers. This was due in part to the openness of the question in that no specific ski resort was provided. When PSTs worked through the task, they spent most of their time collecting data by looking up prices of various ski pass tickets at a local ski resort and debating how often their schedule would allow them to go skiing. One PST, Abe, reflected that this could be an issue when launching the task. He stated,

> I think a potential issue with this lesson could be students finding a place to start, or organizing their data. A few questions to help get them started might be things along the lines of 'what does a best deal actually mean?' or 'in what ways can your money be divided up when choosing to ski/snowboard?' with the hope that this leads them to the understanding that there are different passes.

The instructing PSTs noted that their peers were beginning to use linear equations to model the price of a ski pass as a function of the number of days. One of the instructing PSTs commented that 'this was essentially the math that I was expecting from the question, but we also expected them to take into account renting/not renting equipment, the different ages of the skier, and the cheapest option for different types of skiers.'

The instructing PSTs received valuable feedback to refine their modeling task, mainly, the need to restrict the openness of the problem and limit the amount of assumptions students would need to make. The group was somewhat conflicted with this recommendation in that they were not sure if providing more information and focusing the problem would take away from the idea that modeling is open and creative. By focusing the modeling question to a specific ski resort, the students would have more time to consider other constraints and variables that the instructing PSTs were hoping students would incorporate in their models.

PSTs also discussed the balance between individual and whole class perspectives and the fact that students might not explore a lot of options, only the one that works best for them. One PST, Waldo, stated:

> Students may try and base their final decision on opinion rather than creating a linear equation to determine the best pass. If the students do not have an interest in skiing, then there is a chance that they will claim that none of them are good or will be influenced by a skier/boarder's opinion.

Because there are many factors that one could consider when purchasing a ski pass ticket, the peers also recommended the instructing PSTs include a whole class discussion where the class (versus individual students) decides and discusses what assumptions they are making about skiers, variables, and constraints

Final Modeling Questions
Based on the number of skiing days, what type of Snowy Peaks' [pseudonym for local ski resort] ski pass should you purchase?"

FIG. 7. Ski Pass group's final modeling question.

to include in their models and how these variables and constraints benefit different types of skiers. With these recommendations in mind, the instructing PSTs refined their question as shown in Fig. 7.

One recommendation the instructing PSTs chose *not* to consider related to the relevance of their task to secondary school students. A peer argued that the context of the problem may not resonate with students' lived experiences—many might not be interested in skiing or have the available resources (money and equipment) to be able to ski regularly.

5.2.2. Lesson enactment. The instructing PSTs began their lesson (a snapshot of this group's lesson plan is in the Appendix) by telling students, 'Today, we are going to do a modeling task using the linear equations you have been learning about to decide which type of ski pass to purchase.' They posed their modeling task and asked the class to think about the mathematics they can use to answer this question. In pairs, students attempted to create a diagram to organize the questions they could answer with mathematics and the strategies they could use to do so. The instructing PSTs also encouraged students to incorporate 'one or more important factors' that would influence their purchase. These instructional actions were purposely implemented to guide students to use linear equations in their model and to justify their answer rather than simply choosing one type of pass. PSTs reflected that the task seemed appropriate, and the students employed linear models to compare different passes. Although most of the students were engaged, PSTs discussed how a small group of students were not engaged because the context of the task was not interesting to them. They stated, 'Our motivation aspect was not as motivating as we were hoping for. Very few students expressed interest in skiing, many claiming to have never skied.'

In reflecting on the enactment of the lesson, PSTs acknowledged that anticipating students' strategies and rehearsing guiding questions to ask had helped them to structure the lesson and position themselves as facilitators. They also discussed that providing rationale to the students motivated them to stay engaged with the task. They stated,

> None of the [PSTs] in the room provided answers or strategies for the students. This allowed the students to evaluate the situation and engage in the modeling process. However, guiding questions (many of which were provided in the lesson plan) were used to keep the students' struggle productive and the students willing to participate.

Even though they had anticipated and planned for student strategies, transitions during the lesson were challenging, especially when they were running short on time. Finally, the PSTs discussed that they needed to survey student interest before assuming a task would be personally relevant and engaging to each and every student.

5.2.3. Reflection. The Ski Pass group's reflections illuminated similar conceptions of modeling as the Femur Bone group. In describing mathematical modeling, all PSTs in this group discussed the connection between using mathematics to understand a real-world problem, but only one PST discussed the idea of revision to improve the model. In contrast to the Femur Bone group, all three of PSTs in the Ski Pass group discussed how modeling tasks give purpose and meaning to mathematics through personal engagement. They articulated how students will see relevance in the mathematics they are using and learn that mathematics is a tool that can be applied to their daily lives. In particular, one PST, Kate, stated, '[Modeling] is the real-life stuff of math. You know how most people dislike math? It's usually because it has no relevance to their lives. Mathematical modeling provides that context.' Adding to this, Waldo described that personal engagement, paired with mathematics, allows for connections to be made beyond what is typically covered in lectures. He stated, 'all of this will give the students a chance to use and connect math ideas instead of learn by lecture and notes just to forget after a test.'

Similar to the Femur Bone group, all PSTs in this group discussed the idea of openness and modeling tasks and, in particular, the fear of openness in relation to how they typically know and do math. Kate wrote,

> The openness involved is scary at first. So many [students] used to traditional lectures may need help getting started on the actual [modeling] process. There is no right answer—this is the really scary part, but also the most practical since life is messy and has no single path to solve a problem.

To grapple with openness, two of them specifically discussed that anticipation is an important skill that allowed them to be prepared for different modeling strategies. Hannah stated, 'when implementing a modeling task, it is important to consider many possible solution strategies the students might come up with and be flexible because the students will still come up with something different.'

Lastly, unlike the Femur Bone group, two students in this group also reflected explicitly on the role of the teacher. In a typical mathematics classroom, they described how the teacher often demonstrates particular ways to solve a problem. In a modeling lesson, the teacher must shift from giving ideas to facilitating student exploration and discovery. Kate stated, 'the teacher is there to help and ask questions to develop the model, so students can more clearly see their own ideas, but never to actually say "This is how you solve the problem".'

6. Results

When we consider the research questions in conversation with the modeling competencies and the fluidity of teaching, there is much we can glean from both groups of PSTs. First, there is evidence that both groups developed *theoretical* competencies of what mathematical modeling is and what modeling tasks entail. PSTs demonstrated understanding that modeling involves back and forth between the real-world and the mathematical world, but they did not always allow opportunities for students to discuss and explore different assumptions as they translated a real-world scenario to a mathematical problem. PSTs showed evidence of motivating the context by connecting their task to the real-world and gauging whether a task would be appropriate given particular mathematics content standards. Understanding arose from their experiences as learners and from feedback. Both groups acknowledged that engaging in modeling tasks involves revision, but neither enacted this modeling attribute fully into their modeling lesson.

When considering the *task-related* competencies and how PSTs designed and reasoned through the modeling task, the groups performed differently. The Femur Bone group chose a reasonable task to start, and it was challenging for them to predict and respond to solution strategies outside of their own. The Ski Pass group also chose a reasonable modeling task and was able to think through and propose several solution strategies. We conjecture those difficulties may have happened for two reasons. First, when PSTs are modelers, they often plan and revise one solution and it is not always necessary to understand, respond to or connect multiple approaches. Second, when teaching traditional mathematics content, PSTs typically have a specific problem and/or solution method that is the focus of the lesson. This makes anticipating and guiding student responses easier compared to when teachers enact modeling tasks.

In terms of *teaching* competencies, we saw development in conversation with their *theoretical* competencies. PSTs had to reconcile their understanding of teaching with their experiences as learners. PSTs understood modeling tasks should be open, but they were drawn to also support particular content standards. To reconcile these differences, the Femur Bone group allowed for time to discuss potential mathematical problems at the beginning of the lesson to provide a sense of openness. Whereas the Ski Pass group incorporated guiding questions and practiced taking on the role of facilitator. Both groups enacted different strategies in supporting students at different phases of the modeling process. The Femur Bone group had students work as a whole to develop the model, rather than individual groups. The Ski Pass group was able to support multiple groups of modelers and cycled between small group and whole class discussions. Looking across PSTs, we can see that the development of *teaching* competencies was directly tied to PSTs' experiences with modeling, experiences with teaching, and understanding of task design.

PSTs also developed *diagnostic* competencies and reflection, as both a learner and as a teacher, was key in helping PSTs understand what occurred during the lesson and what changes could be made to support students in the modeling process. The Ski Pass group reflected on their experiences as learners to inform their teaching; as learners, they experienced that modeling takes effort and motivation on the part of the modeler, so they were cognizant of the fact that the task needed to be purposeful and engaging for students. For the Femur Bone group, reflection after instruction was key. They reflected that students were not ready to make the leap between functions and using them to describe a situation and they wanted to provide time for students to wrestle with defining the problem. They realized they needed additional layers of support to bolster students' content knowledge during the modeling process.

When looking across the competencies, PSTs made progress across all four. *Theoretical* competencies developed primarily from time spent working with modeling tasks as learners and receiving feedback from the instructors, CTs, and peers. *Diagnostic* competencies developed from structured time to reflect on their experiences as learners, the development and enactment of the modeling lesson, and changes that could be made to best support student learning. *Task related* and *teaching* competencies were directly related to their perceptions of teaching and learning, and it was sometimes difficult to merge conceptions of modeling with existing conceptions of teaching and task design.

7. Conclusion

We conclude by revisiting our research questions and look across the two groups of PSTs to consider them in relation to one another, identify challenges across modeling competencies, and offer implications for future research.

7.1. How does PSTs' knowledge of teaching modeling develop as they design and implement a modeling task?

Similar to other researchers' work, we found it was important to engage PSTs in the modeling process as learners because it helped them to understand the process and the theoretical competencies involved (Anhalt & Cortez, 2016; Zbiek, 2016). Extending existing research, we found that engaging in the process as learners informed their understanding of the teaching of modeling. Engaging in modeling tasks allowed them to view the task from their students' perspectives to consider what tasks students might find interesting and would foster creativity and motivation.

Although PSTs demonstrated understanding of the modeling process, we saw first-hand that it was challenging for them to assimilate their beliefs on teaching mathematics with their understanding of modeling. This arose in both *task-related* and *teaching* competencies. When considering the *task-related* competency and how PSTs designed and reasoned through their modeling task, there is evidence that it is challenging for PSTs to map out different solution strategies based on different assumptions made by the modeler. We conjecture this is because, as learners, they are not often asked to explore assumptions or justify solution pathways outside of their own. Our research highlights new findings that anticipatory practices can aid them in grappling with openness and multiple solution strategies. PSTs need opportunities to grapple with students' models, explore assumptions made in conjunction with the model, make connections and suggest revisions. This might include mathematics teacher educators developing sample student work for a hypothetical modeling task and providing practice for PSTs to wrestle with understanding them and determining how they would move strategies forward during instruction. We also realized that, when PSTs are modelers, the situation at hand might seem to have endless approaches. In reality, when teaching secondary school students, it is likely there are only a handful of approaches students will use to approach the task. As mathematics teacher educators, we need to help PSTs become more skilled at identifying and grappling with likely approaches and solutions.

When looking across PSTs' *teaching* competencies, our findings provide insight in new directions. Although preparatory activities in the modeling course helped support PSTs in developing and launching an appropriate modeling task; our results indicate that PSTs needed greater supporter in identifying opportunities for students to discuss assumptions and potential approaches to the task. They also needed support navigating the revision, validation, and conclusion aspects of the modeling process, specifically regarding openness. PSTs understood modeling tasks should be open, and they struggled with notions of balancing and teaching content standards while maintaining openness. To restrict the model and limit openness, PSTs made some or all assumptions prior to posing the modeling task rather than allowing students to consider potential assumptions. The context of the Ski Pass task and the way in which it was enacted allowed for more opportunities for students to think through what assumptions might be appropriate (i.e., renting gear, number of available ski days, food, lodging). In contrast, the Femur Bone group did much of the work of exploring assumptions and considering body ratios, prior to enactment. They did not allow students the opportunity to consider the same assumptions that they wrestled with, namely, if this was the best method to determine height or whether a relationship that holds true today would have been true thousands of years ago. Also, in relation to openness, PSTs wrestled with interpreting and connecting students' solutions. Some PSTs experienced openness like a void with infinitely many paths through the model that they could not predict ahead of time. PSTs must feel prepared for openness in order to take on the role of facilitator and help students explore if their model reasonably fits the needs of the situation or needs to be revised. Both of these findings tie back to the importance of the anticipatory work of exploring potential approaches and assumptions the modeler could take and connections that exist between and among students' models. When preparing PSTs as teachers of modeling, they need opportunities to engage in this work, prior to task enactment.

In terms of *diagnostic* competencies and information gleaned from teaching, reflection was key in helping PSTs understand what happened during the lesson and what changes could be made to support students. Reflection helped them to discuss the connections needed to make mathematical progress between topics. Overall, we still have much to learn about *diagnostic* competencies, but reflection created space for PSTs to begin careful consideration of student thinking.

7.2. *How do interactions with instructors, peers, and students shape PSTs' knowledge of teaching modeling?*

For both groups, time with their peers and students was essential in developing their understanding of modeling as teachers. Their peers acted as a sounding board during task development and were helpful in refining tasks. An outside perspective provided different solution strategies that caused them to consider the design of the task, materials to provide students, and the structure of the lesson. This finding illuminates that multiple perspectives are essential in guiding PSTs in the transition from being a modeler to being a teacher of modeling. PSTs benefit from listening to one other and witnessing multiple solution strategies. We also found that PSTs, collectively, were able to generate a wide variety of solutions that helped to prepare them for secondary school students' models. Much of the lesson development and refinement happened prior to meeting with secondary students. This finding indicates that if mathematics teacher educators cannot access local secondary school classrooms, tasking PSTs with a project to design and implement modeling tasks for their peers is likely to also be effective in building modeling competencies. Working with peers did not always go far enough, because peers' work does not capture where school students currently are in their understanding. In the Femur Bone group, although PSTs witnessed multiple approaches, this did not prepare them to help students apply understanding of linear functions in context, something they assumed students would know how to do.

7.3. *Overall, what experiences can be added to a modeling and methods course to prepare PSTs as teachers of modeling?*

Even though we are in a unique position, offering both a full semester modeling and methods of teaching course, there are opportunities that can be drawn from this study and embedded into other universities. First, engaging PSTs in modeling tasks is crucial in developing their understanding of what it means to model and viewing mathematics from the learners' perspective (Doerr, 2007; Anhalt & Cortez, 2016; Zbiek, 2016). Although not all universities can offer a modeling course, modeling tasks can be integrated into existing mathematics content coursework so that PSTs have experiences as learners.

Our findings suggest that discussing the problem, the assumptions and decisions a modeler makes and the multiple solutions that may arise helps to build understanding of what those modeling phases entail. Second, understanding modeling as a learner is not enough to transition tasks into practice. PSTs need targeted rehearsals, enacting and revising their modeling tasks with peers, to develop skills for teaching modeling lessons. These skills entail learning how to interact with students 'in the moment' to identify opportunities in the lesson to elicit student reasoning and the decisions they are making and to draw on that reasoning to further students' progression in the modeling process. These skills also entail learning how to interpret students' models and learn ways to connect different student's approaches. These experiences could be integrated into a methods course with or without practicum experiences.

References

Abassian, A., Safi, F., Bush, S. & Bostic, J. (2019) Five different perspectives on mathematical modeling in mathematics education. *Investig. Math. Learn.* 12, 53–63. https://doi.org/10.1080/19477503.2019.1595360.

Anhalt, C. & Cortez, R. (2016) Developing understanding of mathematical modelling in secondary teacher preparation. *J. Math. Teach. Educ.*, 19, 523–545.

Anhalt, C. O., Cortez, R. & Smith, A. (2017) Mathematical modeling: creating opportunities for participation in mathematics. *Access and Equity: Promoting High Quality Mathematics in Grades 6–8* (A. Fernandes, S. Crespo & M. Civil eds). Reston, VA: National Council of Teachers of Mathematics (NCTM), pp. 105–119.

Blum, W. (2011) Can modelling be taught and learnt? Some answers from empirical research. *Trends in Teaching and Learning of Mathematical Modelling* (G. Kaiser, W. Blum, R. Borromeo Ferri & G. Stillman eds). Dordrecht: Springer, pp. 15–30.

Blum, W. (2015) Quality teaching of mathematical modeling: what do we know, what can we do. *The Proceedings of the 12th International Congress on Mathematical Education* (S. Cho ed). Cham: Springer, pp. 73–96.

Blum, W. & Borromeo Ferri, R. (2009) Mathematical modelling: can it be taught and learnt? *J. Math. Model. Appl.*, 1, 45–48.

Bokova-Guzel, E. (2011) An examination of pre-service mathematics teachers' approaches to construct and solve mathematical modelling problems. *Teach. Math. Appl.*, 30, 19–36.

Borromeo Ferri, R. (2006) Theoretical and empirical differentiations of phases in the modeling process. *ZDM Math. Educ.*, 38, 86–95.

Borromeo Ferri, R. & W. Blum (2010) Mathematical Modelling in Teacher Education - Experiences from a Modelling Seminar. In: Durand-Guerrier, V., Soury-Lavergne, S. & Arzarello, F. (Eds), *CERME-6 - Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*. INRP, Lyon 2010, 2046–2055.

Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Borba, M., Geiger, V., et al. (2014) Mathematical modeling in school education: mathematical, cognitive, curricular, instructional, and teacher education perspectives. *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference on the North American Chapter of the Psychology of Mathematics Education* (P. Liljedahl, C. Nicol, S. Oesterle & D. Allan eds). Vancouver: PME.

Carlson, M. A., Wickstrom, M. H., Burroughs, B. A. & Fulton, E. W. (2016) A case for mathematical modeling in the elementary classroom. *Annual Perspectives in Mathematics Education (APME) 2016: Mathematical Modeling and Modeling Mathematics* (C. Hirsch & A. R. McDuffie eds). Reston, VA: The National Council of Teachers of Mathematics, pp. 121–129.

Cirillo, M., Bartell, T. G. & Wager, A. A. (2016) Teaching mathematics for social justice through mathematical modeling. *Annual Perspectives in Mathematics Education: Mathematical Modeling and Modeling Mathematics* (C. Hirsch ed). Reston, VA: National Council of Teachers of Mathematics, pp. 87–96.

Doerr, H. M. (2007) What knowledge do teachers need for teaching mathematics through applications and modeling? *Modelling and Applications in Mathematics Education: The 14th ICMI Study* (W. Blum, P. L. Galbraith, H.-W. Henn & M. Niss eds). New York, NY: Springer, pp. 69–78.

Doerr, H. M. & Lesh, R. (2011) Models and modelling perspectives on teaching and learning mathematics in the twenty-first century. *Trends in Teaching and Learning of Mathematical Modelling. International Perspectives on Teaching and Learning of Mathematical Modelling* (G. Kaiser, W. Blum, R. Borromeo Ferri & G. Stillman eds). Dordrecht: Springer, pp. 247–268.

Dowling, P. (1999) What does it mean to understand maths as social? *Presented at BERA 1999 as Part of the Symposium*, Online at https://nam10.safelinks.protection.outlook.com/?url=http%3A%2F%2Fwww.pauldowling.me%2Fpublications%2Fbera1999%2Findex.html&data=04%7C01%7Cmegan.wickstrom%40montana.edu%7C287db728c38043870ebb08d8a0358497%7C324aa97a03a644fc91e43846fbced113%7C0%7C0%7C637435499321857944%7CUnknown%7CTWFpbGZsb3d8eyJWIjoiMC4wLjAwMDAiLCJQIjoiV2luMzIiLCJBTiI6Ik1haWwiLCJXVCI6Mn0%3D%7C1000&sdata=Q2ufPd56a8DodXn3qCYDl2A%2FVn%2FcjX6T7GO0lNBfrEw%3D&reserved=0 (Accessed May 2021).

English, L. (2009) *Promoting interdisciplinarity through mathematical modelling. ZDM*, 41, 161–181.

GAIMME: Guidelines for Assessment and Instruction in Mathematical Modeling Education, Second Edition (S. Garfunkel & M. Montgomery eds). COMAP and SIAM, Philadelphia, 2019. View the entire report, available freely online at https://www.siam.org/Publications/Reports/Detail/guidelines-for-assessment-and-instruction-in-mathematical-modeling-education.

Gould, H. (2013) Teachers' conceptions of mathematical modeling. *Doctoral Dissertation*, Available from ProQuest Dissertations and Theses Global. (UMI No. 3560822).

Greer, B., Vershchaffel, L. & Mukhopadhyay, S. (2007) Modelling for life: mathematics and children's experience. *Modelling and Applications in Mathematics Education: The 14th ICMI Study* (W. Blum, P. L. Galbraith, H.-W. Henn & M. Niss eds). New York, NY: Springer, pp. 89–98.

Hammond, M. (2013) The contribution of pragmatism to understanding educational action research: value and consequences. *Educ. Action Res.*, 21, 603–618.

Kaiser, G. (2017) The teaching and learning of mathematical modeling. *Compendium for Research in Mathematics Education* (J. Cai ed). Reston, Virginia: The National Council of Teachers of Mathematics, pp. 267–291.

Kaiser, G. & Sriraman, B. (2006) A global survey of international perspectives on modelling in mathematics education. *ZDM*, 38, 302–310.

Lehrer, R. & Schauble, L. (2007) A developmental approach for supporting the epistemology of modeling. *Modelling and Applications in Mathematics Education: The 14th ICMI Study* (W. Blum, P. L. Galbraith, H.-W. Henn & M. Niss eds). New York, NY: Springer, pp. 155–160.

Lesh, R. & Doerr, H. M. (2003) Foundations of a models and modeling perspective on mathematics teachers, learning, and problem solving. *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching* (R. Lesh & H. M. Doerr eds). Mahwah, NJ: Lawrence Erlbaum Associates, pp. 3–34.

Miles, M. B., Huberman, A. M. & Saldaña, J. (2014) *Qualitative Data Analysis: A Methods Sourcebook*. Thousand Oaks, CA: Sage Publications.

National Governors Association Center for Best Practices & Council of Chief State School Officers (2010) *Common Core State Standards for Mathematics*. Washington, DC: Authors.

Niss, M., Blum, W. & Galbraith, P. (2007) Introduction. *Modelling and Applications in Mathematics Education: The 14th ICMI Study2* (W. Blum, P. Galbraith, H.-W. Henn & M. Niss eds). New York, NY: Springer, pp. 1–32.

Stohlmann, M., Maiorca, C. & Olson, T. (2015) Preservice secondary teachers' conceptions from a mathematical modeling activity and connections to the Common Core State Standards. *Math. Educ. J.*, 24, 21–43.

Turner, E. E., Aguirre, J. M., Foote, M. Q. & Roth McDuffie, A. M. (2018) Learning to leverage mathematical resources of elementary Latinx children through community-based mathematical modeling tasks. *Foregrounding Cultural Ways of Being in Mathematics Teacher Education: Cases From Latinx and Pāsifika Communities*. Presentation as part of a symposium at the annual research conference of the annual meeting of the (M. Civil (Chair)). New York, NY: American Educational Research Association.

Widjaja, W. (2013) Building awareness of mathematical modeling in teacher education: a case study in Indonesia. *Teaching Mathematical Modeling: Connecting to Research and Practice* (G. A. Stillman, G. Kaiser, W. Blum & J. P. Brown eds). Dordrecht: Springer.

Zawojewski, J. S., Lesh, R. & English, L. (2003) A models and modeling perspectives on the role of small group learning activities. *Beyond constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching* (R. A. Lesh & H. Doerr eds). Mahwah, NJ: Erlbaum, pp. 337–358.

Zbiek, R. M. (2016) Supporting teachers' development as modelers and teachers of modelers. *Mathematical modeling and modeling mathematics* (C. R. Hirsch & A. R. McDuffie eds). Reston, VA: The National Council of Teachers of Mathematics, pp. 263–272.

Megan H. Wickstrom is an associate professor of mathematics education in the Department of Mathematical Sciences at Montana State University where she teaches current and future K-12 teachers. Her research focuses on the teaching and learning of mathematical modeling and geometric measurement in K-16 classrooms and understanding best practices when integrating research into classrooms. Email: megan.wickstrom@montana.edu

Elizabeth G. Arnold is an assistant professor of mathematics education in the Department of Mathematics at Colorado State University. Her research centers on the preparation and development of pre-service and in-service K-12 mathematics teachers, with a focus on mathematical modeling, teaching and learning statistics and mathematical knowledge for teaching at the secondary level. Email: liz.arnold@colostate.edu

Appendix

A snapshot of the Femur Bone group’s modeling lesson plan.

<table>
<tr><th colspan="2">Instructional Plan</th></tr>
<tr><th>Teacher Actions</th><th>Student Actions</th></tr>
<tr>
<td>
Motivation (Launch) (3 minutes)

Play this video talking about the new great room
https://www.youtube.com/watch?v=XBKRvcHJUCc
You are an archeologist on the dig into the new room in the Great Pyramid. You recently found a femur bone, how tall is the Egyptian it belonged to?

Lesson Procedures (Explore) (35-40 minutes)

Brainstorm in groups of 3 including the following items:

What tools would you want to use?
What questions do you want to answer to?
What helpful data you need to assist you with solving this problem?
What is your mathematical questions and what math concepts would help answer it?

Share as a class

Ask students what they have brainstormed and make a cloud diagram on the board for all students to see
After students have shared their brainstorming ideas, redirect them to using linear models
Ask students what questions they need answered to create a linear model and discuss as a class
Students hopefully will come to the conclusion that they need possible femur lengths and heights, leading them to the idea of measuring their own femurs for data collection

Data

Have students collect data for individual groups and share with class on board
Make horizontal table on board for student to put in their input

Time with data

Direct students to use the graph paper to graph their data (plot data points)
Have <u>students decide</u> the number of data points they want to use
Sketch a <u>STRAIGHT</u> line that best fits their data points
Remind students to create a linear model, which means a model in the form y=mx +b

Have students give models to the instructors to be presented

Talk about the differences between models
First have students make observations about what they see in each model, hopefully differences will occur
Then have students discuss how they created their model

</td>
<td>

Throughout the entire lesson, walk around the room to answer questions, probe students deeper, and make sure that they are on task
Students should brainstorm ways to figure out how tall the Egyptian would have been in their group, and hopefully realize they need to gather data
Don’t let students use technology in the brainstorming stage
Students will use the tape measures to measure each other’s femurs/heights
Students will only measure their group members, and record the lengths and heights on the board so that the whole class has access to everyone’s data
Students should record their group’s data in the horizontal table written on the board (you don’t want to push students to create their model one way)
Misconception - students might be tempted to connect each of their data points. This is when you remind them of a linear model.

</td>
</tr>
</table>

A snapshot of the Ski Pass group's lesson plan.

Instructional Plan	
Teacher Actions	**Student Actions**
Motivation (Launch) – (7-10 min.) Today, we are going to do a modeling task using thelinear equations you have been learning about to decide which type of ski pass to purchase. We will beworking in groups and researching using the internet,so your full attention will be needed in order to get a solution by the end of the period. Cell phones will notbe necessary to solve this problem, so make sure to put them away or put them in the caddy so they will not be a distraction" Show youtube video of skiing: https://youtu.be/8ANDO-zT3hq Raise the question: "At Snowy Peaks, what is the best pass for you to buy?" Let the class discuss the different solution options. Ask the question: What math can you use to solve thisproblem? Conduct a think-pair-share to discuss solution options. Guide the students to analyze the cost of the season pass after students have a chance to share their thoughts. **Transition:** Explain to the students that they will be working in teams to design their own models for buying a ski pass based on the factors their group thinks are the most important. Break the students intogroups of four (the groups they sit with). **Lesson Procedures (Explore) – (30 min.)** Sending one group up at a time, have each student retrieve a Chromebook from the mobile computer cart. Direct the students to the Snowy Peaks Website. Each group will assign the roles of facilitator, devil's advocate, recorder, messenger, and any other needed roles, among their group members. Each member will contribute to the brainstorming process.	**Motivation** *Expected Student Behaviors/Responses:* Students may get excited about the skiing video. It may be hard to get the class to quietdown again to focus on the mathematics. They may become distracted and watch more videos during the lesson procedure portion. Students may be confused by the question from the start and need help getting their feetunder them. Students may also choose the wrong approach that will not allow them to solve theproblem with linear equations. Students may choose a single pass option and call it good, never addressing the mathematics or creating linear equations. *Suggestions for Instructor:* Pose a few questions that the students can consider/work off of. Ex: How frequently do you go to BridgerBowl? Keep an eye on the student's progress and strategies to make sure they are on the rightpath for linear equations. Ask the students how their answer would compare mathematically to another potentialproblem. Have them try and explain their argument using linear equations. **Lesson Procedures** *Expected Student Behaviors/Responses:* Students may have disagreements within groups about what factors they will use in their model due to different opinions on skiing/snowboarding.